USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 2.4, we will be asked to express linear systems as the **matrix equation AX = B**, where A, X, and B are matrices.

- Matrix A is called the **coefficient matrix.**
- Matrix X is a matrix with 1 column that contains the variables.
- Matrix B is a matrix with 1 column that contains the constants.

◆*Example 11* Verify that the system of two linear equations with two unknowns:

$$ax + by = h$$
$$cx + dy = k$$

can be written as AX = B, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and } B = \begin{bmatrix} h \\ k \end{bmatrix}$$

Solution: If we multiply the matrices A and X, we get

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If AX = B then

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$ax + by = h$$
$$cx + dy = k$$

◆*Example 12* Express the following system as a matrix equation in the form AX = B.

$$2x + 3y - 4z = 5$$
$$3x + 4y - 5z = 6$$
$$5x \qquad - 6z = 7$$

Solution: This system of equations can be expressed in the form AX = B as shown below.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations.

As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom.

We state the Gauss-Jordan method as follows.

Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

◆***Example 5*** Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x + y + 2z &= 10 \\ x + 2y + z &= 8 \\ 3x + y - z &= 2 \end{aligned}$$

Solution: We write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right]$$ we interchanged row 1(R1) and row 2(R2)

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1 + R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1 + R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad R2 \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2 + R1 \text{ and } 5R2 + R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -R3 + R1$$

Clearly, the solution reads $x = 1$, $y = 2$, and $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

◆***Example 2*** Solve the following system of equations.

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ 3x + 4y - 2z &= 9 \\ 5x + 7y - 6z &= 20 \end{aligned}$$

Solution: We enter the following augmented matrix in the calculator.

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

Now by pressing the key to obtain the reduced row-echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row states that $0x + 0y + 0z = 1$. But the left side of the equation is equal to 0. So this last row states $0 = 1$, which is a contradiction, a false statement.

This bottom row indicates that the system is inconsistent; therefore, there is no solution.

◆***Example 3*** Solve the following system of equations.

$$\begin{aligned} x + y &= 7 \\ x + y &= 7 \end{aligned}$$

Solution: The problem clearly asks for the intersection of two lines that are the same; that is, the lines coincide. This means the lines intersect at an infinite number of points.

A few intersection points are listed as follows: (3, 4), (5, 2), (–1, 8), (–6, 13) etc. However, when a system has an infinite number of solutions, the solution is often expressed in the parametric form. This can be accomplished by assigning an arbitrary constant, t, to one of the variables, and then solving for the remaining variables. Therefore, if we let $y = t$, then $x = 7 - t$. Or we can say all ordered pairs of the form $(7 - t, t)$ satisfy the given system of equations.

Alternatively, while solving the Gauss-Jordan method, we will get the reduced row-echelon form given below.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

The row of all zeros, can simply be ignored. This row says $0x + 0y = 0$; it provides no further information about the values of x and y that solve this system.

This leaves us with only one equation but two variables. And whenever there are more variables than the equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as above.

Parametric Solution: $x = 7 - t$, $y = t$.

◆***Example 4*** Solve the following system of equations.

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 3x + 2y \quad &= 5 \end{aligned}$$

Solution: The augmented matrix and the reduced row-echelon form are given below.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right]$$ Augmented Matrix for this system

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Reduced Row Echelon Form

Since the last equation dropped out, we are left with two equations and three variables. This means the system has infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$.

The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$.

And now if we let $z = t$, the parametric solution is expressed as follows:

Parametric Solution: $\mathbf{x = 1 + 2t, \quad y = 1 - 3t, \quad z = t.}$

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- if we let $t = 2$, we have the solution $x = 5$, $y = -5$, $z = 2$: $(5, -5, 2)$
- if we let $t = 0$, we have the solution $x = 1$, $y = 1$, $z = 0$: $(1, 1, 0)$.

◆***Example 5*** Solve the following system of equations.

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution: The reduced row-echelon form is given below.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.

If we let $z = t$ and let $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$.

We rewrite the **parametric solution :** $\mathbf{x = 5 - 2s + 3t, \quad y = s, \quad z = t.}$

2.4 Inverse Matrices

In this section you will learn to:

1. *Find the inverse of a matrix, if it exists.*
2. *Use inverses to solve linear systems.*

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition of an Inverse: An $n \times n$ matrix has an inverse if there exists a matrix B such that $AB = BA = I_n$, where I_n is an $n \times n$ identity matrix. The inverse of a matrix A, if it exists, is denoted by the symbol A^{-1}.

◆***Example 1*** Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution: The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2: I_2,

$$AB = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

◆***Example 2*** Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution: Suppose A has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$$3a + c = 1 \qquad 3b + d = 0$$
$$5a + 2c = 0 \qquad 5b + 2d = 1$$

Solving this system, we get: $a = 2$ $\quad b = -1$ $\quad c = -5$ $\quad d = 3$

Therefore, the inverse of the matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$3a + c = 1 \qquad 3b + d = 0$$
$$5a + 2c = 0 \qquad 5b + 2d = 1$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

The matrix on the right side of the vertical line is the A^{-1} matrix.

What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix. We list the steps, as follows:

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[\ A \mid I_n\ ]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[\ I_n \mid B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

◆***Example 3*** Given the matrix A below, find its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution: We write the augmented matrix as follows.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method.

Multiplying the first row by –2 and adding it to the second row, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by –2. The result is

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) Add the second row to first, 2) Add -5 times the second row to the third. And we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 results in

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right],$$

Multiply the third row by 1/2 and add it to the second.
Also, multiply the third row by –1/2 and add it to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A is $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

◆***Example 4*** Solve the following equation: $\frac{2}{3}x = 4$

Solution: To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$ which happens to be $\frac{3}{2}$. We get

$$\frac{3}{2} \cdot \frac{2}{3} x = 4 \cdot \frac{3}{2}$$

$$x = 6.$$

We use the Example 4 as an analogy to show how linear systems of the form AX = B are solved.

To solve a linear system, we first write the system in the matrix equation AX = B, where A is the coefficient matrix, X the matrix of variables, and B the matrix of constant terms.
We then multiply both sides of this equation by the multiplicative inverse of the matrix A.

Consider the following example.

◆***Example 5*** Solve the following system

$$3x + y = 3$$
$$5x + 2y = 4$$

Solution: To solve the above equation, first we express the system as

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation AX = B by A^{-1}.
Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example 2, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Multiplying both sides by A^{-1}, we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, x = 2, and y = –3.

◆***Example 6*** Solve the following system:

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solution: To solve the above equation, we write the system in matrix form AX = B as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of A. From Example 3, $A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation AX = B on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[\, A \mid I_n \,]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[\, I_n \mid B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

The Method for Solving a System of Equations When a Unique Solution Exists

1. Express the system in the matrix equation AX = B.
2. To solve the equation AX = B, we multiply on both sides by A^{-1}.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \quad \text{where I is the identity matrix} \end{aligned}$$